

# Extended Hydrodynamics from Enskog's Equation for a Two-Dimensional System General Formalism

Hideaki Ugawa<sup>1</sup> and Patricio Cordero<sup>1</sup>

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Balance equations are derived from Enskog's kinetic equation for a two-dimensional system of hard disks using Grad's moment expansion method. This set of equations constitute an extended hydrodynamics for moderately dense bi-dimensional fluids. The set of independent hydrodynamic fields in the present formulations are: density, velocity, temperature *and also*—following Grad's original idea—the symmetric and traceless pressure tensor  $p_{ij}$  and the heat flux vector  $\mathbf{q}^k$ . An approximation scheme similar in spirit to one made by Grad in his original work is made. Once the hydrodynamics is derived it is used to discuss the nature of a simple one-dimensional heat conduction problem. It is shown that, not too far from equilibrium, the nonequilibrium pressure in this case only depends on the density, temperature and heat flux vector.

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## 1. INTRODUCTION

In 1921 Enskog introduced a kinetic theory for dense gases<sup>(1)</sup> which is known to yield a very good approximate description of the behavior of gases, particularly transport phenomena, as in Ref. 2 and references therein. Today Enskog's original theory is known as the standard Enskog theory (SET)<sup>(2,4)</sup> because after the pioneer work of van Beijeren and Ernst<sup>(5)</sup> there are several new versions of Enskog's theory collectively called revised Enskog's theory (RET).<sup>(6)</sup> Among the latter there are versions that have been extended to describe condensed matter.<sup>(7)</sup> To Navier-Stokes level both SET and RET lead to the same results,<sup>(5,8)</sup> whether or not an external force is present. For SET and RET, using approximations in Sec. 5, the same hydrodynamic equations are obtained.

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<sup>1</sup> Departamento de Física, FCFM, Universidad de Chile, Santiago, Chile; e-mail: pcordero@uchile.cl.

Once a kinetic equation is given it is possible to derive hydrodynamic like equations, thus reducing the degrees of freedom from those of the velocity distribution function  $f(\mathbf{r}, \mathbf{c}, t)$  to the degrees of freedom of a finite set of hydrodynamic fields. Of these methods we mention the Chapman–Enskog method<sup>(3,9,10)</sup> and Grad’s moment expansion method.<sup>(11,12)</sup> The first is widely used and illustrated even in textbooks. Much less attention has been paid to Grad’s method which is our central tool in this paper. Both Chapman–Enskog and Grad’s methods are truncated expansions.

Perhaps the first effort to obtaining hydrodynamic equations (balance equations) from Enskog’s theory using Grad’s moment expansion methods was by Schmidt *et al.*<sup>(13)</sup> in 1981. They did not start from the local distribution function but from a Maxwell distribution function as kernel, totally linearized their hydrodynamic equations, that is, completely eliminated the products of differences between the hydrodynamic variables associated to the equilibrium state and the inhomogeneous nonequilibrium state. From this they obtained approximate hydrodynamic equations such that the conservation laws for mass, moment and energy are not exactly satisfied. They also obtained expressions for the transport coefficients (viscosity and heat conductivity) and the thermal pressure.

In 1988, Kremer and Rosa<sup>(14)</sup> obtained hydrodynamic equations from the local equilibrium distribution function as kernel linearizing the collision integral in Enskog’s equation. In particular they introduced an approximation which completely eliminates the second order terms of the collision integral. They derived sound dispersion relations for monoatomic gases using normal mode analysis.

In 1991, Marques and Kremer<sup>(15)</sup> developed Kremer and Rosa’ work<sup>(14)</sup> and obtained linearized hydrodynamic equations involving the second order terms of the collision integral. Consequently their equations are less approximate than in Ref. 13 although in the bi-dimensional case, eliminating the products between the hydrodynamic variables which are deviations from the value at equilibrium, their equations coincide with our results. Furthermore they obtained linearized Burnett equations for monoatomic gases where they eliminate third order terms from the collision integral.

In 1996, Rangel-Huerta and Velasco<sup>(16)</sup> obtained the hydrodynamic equations from the local equilibrium distribution function as kernel with partial linearization of the collision integral in Enskog’s equation. They also eliminated collision terms as in Ref. 14 and obtained the extended equations as in Ref. 15: in the second cited paper of Ref. 16 Grad’s distribution function without approximation is used, whereas in Ref. 15 the authors use a Maxwellian distribution function to get the second order collision term. Therefore, the terms of the second order spatial derivative of the pressure tensor and heat flow vector are added in the balance equations for the pressure tensor and heat flow vector. They derived generalized transport coefficients with the methods used widely in generalized hydrodynamics.<sup>(17)</sup> In 1997, Ref. 18 develops the extension of Grad’s moment

method for the RET, and in 1998, obtained hydrodynamic equations which are applicable only to a system undergoing uniform shear flow.<sup>(19)</sup> In 2001, Rangel-Huerta and Velasco<sup>(20)</sup> eliminated the second order terms of the spatial derivative of the pressure tensor and heat flow vector from the balance equations for the pressure tensor and heat flow vector of the former results<sup>(16)</sup> and obtained a similar but extended hydrodynamic equations similar to Ref. 15 nevertheless, get better agreement with molecular dynamic simulations.<sup>(21)</sup>

The present article provides extended hydrodynamic equations derived from Enskog's equation using Grad's moment expansion method in the bi-dimensional case. They are more complete than a linear approximation but still they are the result of an approximation scheme that we later explain. As far as we know this is the first time that Grad's method used to obtain extended hydrodynamic equations from Enskog's equation has been published beyond the a linear approximation in two or three dimensions.

In addition, we will apply this hydrodynamic equations to a one dimensional steady state heat conduction case. There are many studies on this theme: for example, the experimental investigation in Ref. 22 and the theoretical one in Refs. 23, 24 Gross and Ziering<sup>(25)</sup> investigated it starting from Boltzmann's equation with Grad's moment method. To our knowledge, no one has investigated it starting from Enskog's equation using Grad's moment method. Recently Kim and Hayakawa<sup>(26)</sup> investigated it starting from Boltzmann's equation using the Chapman–Enskog method for hard core particles, Maxwellian ones and BGK's ones<sup>(27)</sup> and Hayakawa *et al.*<sup>(28)</sup> investigated it for hard disks using molecular-dynamic simulations. Furthermore, they tried a test of the nonequilibrium steady state thermodynamics (SST) proposed by Sasa and Tasaki<sup>(29)</sup> and criticized it. The state defined in Ref. 29 for a gas in a nonequilibrium steady state is a one-dimensional heat conductive configuration in contact with an equilibrium state through a special porous wall (called the perfect  $\mu$  wall.<sup>(29)</sup>) There is a nontrivial pressure difference between the equilibrium and nonequilibrium state parallel to the direction of the heat flow. We have considered this phenomenon with our hydrodynamic equations in Ref. 30.

Parenthetically we mention that it is meaningful to check, using kinetic theory, whether one of the main assumptions of SST<sup>(29)</sup>—the nonequilibrium steady state pressure—is a functional of the density, temperature and heat flow. We find that, in fact, it is correct provided the system is not too far from equilibrium. This is consistent with the range predicted in Ref. 29 for this nontrivial pressure difference to appear. Note that one of the useful merits of Grad's moment method is that one can easily find relations between nonequilibrium thermodynamic variables.

The organization of this paper is as follows. In Sec. 2, Grad's moment expansion method is introduced. Section 3 is about Enskog's equation. In Sec. 4 we formulate the macroscopic balance equations. In Sec. 5 the simplified collision contribution used in this paper is constructed showing that, for the approximations

introduced here, there is no difference between the macroscopic balance equations obtained by RET and SET. In Sec. 6 the hydrodynamic equations that our procedure yields are presented and the collision frequency coefficient  $\chi$  used in this paper are introduced. In Sec. 7 we apply our hydrodynamic equations to a simple heat conduction case in which there is a steady heat current between two parallel plates at slightly different temperatures. The closed expression of the pressure only depends on the hydrodynamic fields: density, temperature and heat vector. A final remarks on this example are given. In Sec. 8 we summarize our results.

## 2. THE VELOCITY DISTRIBUTION FUNCTION AND ITS EXPANSION

To define dimensionless variables  $A$  from the physical variables  $\bar{A}$  we choose the mass of the particles to be the unit of mass, and the temperature is measured in energy units so that Boltzmann's constant is 1. Further, we take a reference distance  $L$  and a reference temperature  $T_0$  so that the dimensionless coordinates  $x$ , velocities  $c$  and time are such that

$$x = \bar{x}/L, \quad c = \bar{c}/\sqrt{T_0}, \quad t = \frac{\sqrt{T_0}}{L} \bar{t}.$$

Assuming that the system is in a rectangular box of size  $L_x \times L_y$  the hydrodynamic fields can be rescaled defining dimensionless fields as follows,

$$\begin{aligned} n &= \frac{L_x L_y}{N} \bar{n} \quad \text{number density,} \\ \mathbf{v} &= \frac{1}{\sqrt{T_0}} \bar{\mathbf{v}} \quad \text{hydrodynamic velocity,} \\ T &= \frac{1}{T_0} \bar{T} \quad \text{temperature,} \\ P_{ij} &= \frac{L_x L_y}{N T_0} \bar{P}_{ij} \quad \text{pressure tensor,} \\ \mathbf{q} &= \frac{L_x L_y}{N T_0^{3/2}} \bar{\mathbf{q}} \quad \text{energy flux,} \\ \mathbf{F} &= \frac{L}{T_0} \bar{\mathbf{F}} \quad \text{external force,} \end{aligned} \tag{1}$$

The distribution function  $\bar{f}$  is replaced by a dimensionless distribution function  $f$ ,

$$f = \frac{L_x L_y T_0}{N} \bar{f}.$$

In the following sections some auxiliary dimensionless quantities will be used

$$\begin{aligned} \alpha_y &= \frac{L_x}{L}, \\ \alpha_x &= \frac{L_y}{L}, \\ \rho_0 &= \frac{N}{L_x L_y} \frac{\pi \sigma^2}{4} = \frac{\pi N \delta^2}{4 \alpha_x \alpha_y}, \\ \text{Kn} &= \frac{8\sqrt{2} \ell}{\pi L}, \quad \text{Knudsen number,} \\ \delta &= \frac{\sigma}{L} = \text{Kn} \rho_0 \end{aligned} \tag{2}$$

where  $\sigma$  is the disk's diameter,  $N$  is the number of disks and  $\ell = \frac{\pi \sigma}{8\sqrt{2}\rho_0}$  is the mean free path at equilibrium. In these units, for example, the free flight time for Boltzmann's gases at equilibrium temperature  $T_0$  is  $\frac{\sqrt{\pi \text{Kn}}}{8}$ .  $L$  is any macroscopic distance that we take as relevant depending on the particular problem to be analyzed. For the purposes of the present paper we will typically assume that  $L$  is of the order of  $L_y$ .

Following Grad, we make a formal eight moments expansion of the distribution function<sup>(11)</sup>

$$f(\mathbf{r}, \mathbf{c}, t) = f_M \Phi_G, \tag{3}$$

where

$$f_M = n(\mathbf{r}, t) \left[ \frac{1}{2\pi T(\mathbf{r}, t)} \right] \exp \left[ -\frac{C(\mathbf{r}, t)^2}{2T(\mathbf{r}, t)} \right]. \tag{4}$$

In Grad's method the factor  $\Phi_G$  is written as an expansion in Hermite polynomials of the peculiar velocity  $\mathbf{C} \equiv \mathbf{c} - \mathbf{v}(\mathbf{r}, t)$ ,

$$\Phi_G = \sum_{n=0}^{\infty} a_i^{(n)}(\mathbf{r}, t) H_i^{(n)}(\mathbf{C}), \tag{5}$$

where  $\mathbf{c}$  is the molecular velocity and where  $H_i^{(n)}$  is a tensor with  $n$  subscripts,  $i = (i_1 \dots i_n)$  as well a polynomial of  $n$ -th degree in the components of  $\mathbf{C}$ . Also the coefficients  $a_i^{(n)}$  are tensors of order  $n$ . The eight moments expansion corresponds to truncating the expansion in Eq. (5) up to  $n = 3$ . In dimension 3 this leads to the well known 13 moments methods used by Grad himself working upon Boltzmann's equation. It is widely used.

Grad's method consists of first replacing  $f$  given by Eq. (3) but with the truncated expression for  $\Phi_G$ , in the kinetic equation (Boltzmann's or Enskog's), and

then projecting the kinetic equation to the different Hermite polynomials used in the truncated expansion. This procedure—devised by Grad—leads to obtaining as many balance equations as polynomials were included in the truncated expansion for  $\Phi_G$ .

The Hermite polynomials satisfy the orthogonality relations

$$\frac{1}{2\pi T} \int_{-\infty}^{\infty} H_a^m H_b^n \exp\left[-\frac{C^2}{2T}\right] d\mathbf{C} = n! \delta_{mn} \delta_{ab}, \quad (6)$$

and they are, for example,

$$H^0 = 1, \quad (7)$$

$$H_i^1 = \frac{C_i}{\sqrt{T}}, \quad (8)$$

$$H_{ij}^2 = \frac{C_i C_j}{T} - \delta_{ij}, \quad (9)$$

$$H_{ijk}^3 = \frac{C_i C_j C_k}{T^{\frac{3}{2}}} - \frac{(C_i \delta_{jk} + C_j \delta_{ki} + C_k \delta_{ij})}{\sqrt{T}} \quad (10)$$

where  $i, j = 1, 2$ . Instead of using the full set of third order Hermite polynomial  $H_{ijk}^3$  we use—as Grad did—the corresponding contracted Hermite polynomials  $H_i^3$  defined by

$$H_i^3 = \frac{C_i}{\sqrt{T}} \left( \frac{C^2}{T} - 4 \right). \quad (11)$$

The coefficients  $a_i^n$  in Eq. (5) can be expressed in terms of the hydrodynamic fields  $n(\mathbf{r}, t)$ ,  $T(\mathbf{r}, t)$ , and the flux vectors associated with the velocity and the energy, called the kinetic part of pressure tensor and the energy flux:  $P_{ij}^k(\mathbf{r}, t)$ ,  $q_i^k(\mathbf{r}, t)$  according to the following sum rules,

$$\int f d\mathbf{c} = n(\mathbf{r}, t), \quad (12)$$

$$\int \mathbf{C}(\mathbf{r}, t) f d\mathbf{c} = 0, \quad (13)$$

$$\int \frac{1}{2} \mathbf{C}(\mathbf{r}, t)^2 f d\mathbf{c} = n(\mathbf{r}, t) T(\mathbf{r}, t), \quad (14)$$

$$\int C_i(\mathbf{r}, t) C_j(\mathbf{r}, t) f d\mathbf{c} = P_{ij}^k, \quad (15)$$

$$\int \frac{1}{2} \mathbf{C}(\mathbf{r}, t) C^2(\mathbf{r}, t) f d\mathbf{c} = \mathbf{q}^k(\mathbf{r}, t), \quad (16)$$

where  $p_{ij}$  is the traceless part of  $P_{ij}^k$ ,

$$P_{ij}^k = n(\mathbf{r}, t)T(\mathbf{r}, t)\delta_{ij} + p_{ij}(\mathbf{r}, t).$$

The factor  $\Phi_G$  for the eight moments method turns out to be

$$\Phi_G = 1 + \frac{1}{2nT^2}p_{i,j}C_iC_j + \frac{1}{nT^2}\left[\frac{C^2}{4T} - 1\right]\mathbf{C} \cdot \mathbf{q}^k. \quad (17)$$

We refer to the “kinetic part” of the fluxes to stress that kinetic theory, in principle, includes contributions to the fluxes associated to the interaction between the particles.

### 3. ENSKOG'S EQUATION

Enskog's equation is

$$\begin{aligned} & \left[ \frac{\partial}{\partial t} + \mathbf{c}_1 \cdot \nabla_1 + \mathbf{F} \cdot \nabla_{\mathbf{c}_1} \right] f(\mathbf{r}_1, \mathbf{c}_1, t) \\ &= \frac{4}{\pi K n} \int [\chi(\mathbf{r}_1, \mathbf{r}_1 + \delta\mathbf{k} | n) f(\mathbf{r}_1, \mathbf{c}'_1, t) f(\mathbf{r}_1 + \delta\mathbf{k}, \mathbf{c}'_2, t) \\ & \quad - \chi(\mathbf{r}_1, \mathbf{r}_1 - \delta\mathbf{k} | n) f(\mathbf{r}_1, \mathbf{c}_1, t) f(\mathbf{r}_1 - \delta\mathbf{k}, \mathbf{c}_2, t)] (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_2 \end{aligned} \quad (18)$$

where  $\mathbf{k}$  is the unit vector from the disk centered at  $\mathbf{r}_1 + \delta\mathbf{k}$  to the disk with center at  $\mathbf{r}_1$  upon collision and is integrated over all the unit vectors while  $\mathbf{g} = \mathbf{c}_2 - \mathbf{c}_1$  and  $\theta_{\mathbf{k}}$  is the Heaviside function  $\theta_{\mathbf{k}} = \theta(\mathbf{g} \cdot \mathbf{k})$  and where the two velocities of the disks after collision are  $\mathbf{c}'_1 = \mathbf{c}_1 + (\mathbf{g} \cdot \mathbf{k})\mathbf{k}$  and  $\mathbf{c}'_2 = \mathbf{c}_2 - (\mathbf{g} \cdot \mathbf{k})\mathbf{k}$ , respectively.

The collision frequency  $\chi(\mathbf{r}_1, \mathbf{r}_1 \pm \delta\mathbf{k} | n)$  is the pair distribution function of two hard disks at contact and  $n$  is the number density defined by Eq. (12). This  $\chi$  is given by the procedure of statistical mechanics, that is, by the virial expansions of the pair distribution function and of course its expression depends on whether one uses SET or RET.

Concretely, with  $\mathbf{r}' = \mathbf{r} + \delta\mathbf{k}$ ,

$$\begin{aligned} \chi^{SET}(\mathbf{r}, \mathbf{r}' | n) &= 1 + n \left( \frac{1}{2}(\mathbf{r} + \mathbf{r}') \right) \int V(\mathbf{r} | \mathbf{r}_3) d\mathbf{r}_3 + \frac{1}{2!} n^2 \left( \frac{1}{2}(\mathbf{r} + \mathbf{r}') \right) \\ & \quad \times \int V(\mathbf{r}, \mathbf{r}' | \mathbf{r}_3 \mathbf{r}_4) d\mathbf{r}_3 d\mathbf{r}_4 + \dots \\ &= \chi_c + \frac{\delta}{2} \mathbf{k} \cdot \nabla \chi_c + \frac{\delta^2}{8} \mathbf{k} \mathbf{k} : \nabla \nabla \chi_c + \dots \end{aligned} \quad (19)$$

$$\begin{aligned}
\chi^{RET}(\mathbf{r}, \mathbf{r}'|n) &= 1 + \int n(\mathbf{r}_3)V(\mathbf{r}, \mathbf{r}'|\mathbf{r}_3) d\mathbf{r}_3 + \frac{1}{2!} \\
&\quad \times \int n(\mathbf{r}_3)n(\mathbf{r}_4)V(\mathbf{r}, \mathbf{r}'|\mathbf{r}_3\mathbf{r}_4) d\mathbf{r}_3 d\mathbf{r}_4 + \dots \\
&= \chi_c + \frac{\delta}{2} \frac{\partial \chi_c}{\partial n} (\mathbf{k} \cdot \nabla n) + \frac{\delta^2}{8} \frac{\partial^2 \chi_c}{\partial n^2} (\mathbf{k} \cdot \nabla n)(\mathbf{k} \cdot \nabla n) + \dots \quad (20)
\end{aligned}$$

where

$$\chi_c = 1 + n(\mathbf{r}) \int V(\mathbf{r}, \mathbf{r}'|\mathbf{r}_3) d\mathbf{r}_3 + \frac{n^2(\mathbf{r})}{2!} \int V(\mathbf{r}, \mathbf{r}'|\mathbf{r}_3\mathbf{r}_4) d\mathbf{r}_3 d\mathbf{r}_4 + \dots \quad (21)$$

and where the usual Hushimi V-function is introduced.<sup>(8,31)</sup>

Expanding the right hand side in powers of  $\delta$  up to second order following the Appendix of Ref. 8,

$$\text{rhs}^{SET} = J_0^{SET} + \delta J_1^{SET} + \delta^2 J_2^{SET}, \quad (22)$$

$$\text{rhs}^{RET} = J_0^{RET} + \delta J_1^{RET} + \delta^2 J_2^{RET}, \quad (23)$$

where

$$\begin{aligned}
J_0^{SET} &= J_0^{RET} = J_0, \\
J_1^{SET} &= J_1^{RET} = J_{11} + J_{12}, \\
J_2^{SET} &= J_{21} + J_{22} + J_{23}, \\
J_2^{RET} &= J_2^{SET} + J_{24}, \quad (24)
\end{aligned}$$

and

$$\begin{aligned}
J_0 &= \frac{4}{\pi \text{Kn}} \chi_c \int (f'_1 f'_2 - f_1 f_2) (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_2, \\
J_{11} &= \frac{4}{\pi \text{Kn}} \chi_c \int \mathbf{k} \cdot (f'_1 \nabla f'_2 + f_1 \nabla f_2) (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_2, \\
J_{12} &= \frac{2}{\pi \text{Kn}} \int (\mathbf{k} \cdot \nabla \chi_c) (f'_1 f'_2 + f_1 f_2) (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_2, \\
J_{21} &= \frac{2}{\pi \text{Kn}} \chi_c \int \mathbf{k} \mathbf{k} : (f'_1 \nabla \nabla f'_2 - f_1 \nabla \nabla f_2) (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_2, \quad (25) \\
J_{22} &= \frac{2}{\pi \text{Kn}} \int (\mathbf{k} \cdot \nabla \chi_c) \mathbf{k} (f'_1 \nabla f'_2 - f_1 \nabla f_2) (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_2, \\
J_{23} &= \frac{1}{2\pi \text{Kn}} \int (\mathbf{k} \mathbf{k} : \nabla \nabla \chi_c) (f'_1 f'_2 - f_1 f_2) (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_2, \\
J_{24} &= -\frac{1}{2\pi \text{Kn}} \frac{\partial \chi_c}{\partial n} \int (\mathbf{k} \mathbf{k} : \nabla \nabla n(\mathbf{r}, t)) (f'_1 f'_2 - f_1 f_2) (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_2
\end{aligned}$$



where the following abbreviations:  $f_1 = f(\mathbf{r}_1, \mathbf{c}_1, t)$ ,  $f'_1 = f(\mathbf{r}_1, \mathbf{c}'_1, t)$ ,  $f_2 = f(\mathbf{r}_1, \mathbf{c}_2, t)$  and  $f'_2 = f(\mathbf{r}_1, \mathbf{c}'_2, t)$  are introduced. Here  $\chi_c$  is concretely given by the virial expansions,<sup>(8)</sup> but, as seen below, we do not use such an expression.

#### 4. THE BALANCE EQUATIONS

The macroscopic balance equations are derived following Grad's prescription, successively projecting the kinetic equation to the different Hermite polynomials being used.

If Enskog's equation is multiplied by a function  $\phi_1$  of the peculiar velocity  $\mathbf{C}_1 = \mathbf{c}_1 - \mathbf{v}$  and integrating over  $\mathbf{c}_1$  yields

$$\begin{aligned} & \int \phi_1 \frac{\partial}{\partial t} f_1 d\mathbf{c}_1 + \int \phi_1 \mathbf{c}_1 \cdot \nabla f_1 d\mathbf{c}_1 + \int \phi_1 \mathbf{F} \cdot \nabla_{\mathbf{c}_1} f_1 d\mathbf{c}_1 \\ & = I \\ & = I_0^{SET} + \delta I_1^{SET} + \delta^2 I_2^{SET} \quad (\text{for SET}) \\ & = I_0^{RET} + \delta I_1^{RET} + \delta^2 I_2^{RET} \quad (\text{for RET}) \end{aligned} \quad (26)$$

where

$$I_i = \int \phi_1 J_i d\mathbf{c}_1 \quad (i = 0, 1, 2). \quad (27)$$

The balance equations of our interest, in particular the basic conservation equations (mass, momentum and energy), are obtained multiplying Enskog's equation by  $\phi = 1$ ,  $\mathbf{c}_1$ ,  $\frac{1}{2}C_1^2$ . In such cases  $\phi_1 + \phi_2 = \phi'_1 + \phi'_2$  is true (prime indicates post collision velocities) so that

$$I_0^{SET} = I_0^{RET} = 0, \quad (28)$$

and

$$I_1^{SET} = I_1^{RET} = \frac{2}{\pi K\mathfrak{N}} \nabla \cdot \int \chi_c (\phi_1 - \phi'_1) f_1 f_2 (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} \mathbf{k} d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}_2, \quad (29)$$

$$\begin{aligned} I_2^{SET} = I_2^{RET} & = \frac{1}{\pi K\mathfrak{N}} \nabla \cdot \int \chi_c (\phi_1 - \phi'_1) \\ & \times \left[ \mathbf{k} \cdot f_1 f_2 \nabla \log \frac{f_1}{f_2} \right] (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} \mathbf{k} d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}_2. \end{aligned} \quad (30)$$

In the case when the external force is uniform the general conservation equation may be written as

$$\frac{\partial}{\partial t} \int \phi_1 f_1 d\mathbf{c}_1 + \nabla \cdot \int \phi_1 \mathbf{c}_1 f_1 d\mathbf{c}_1 - \mathbf{F} \cdot \int \phi_1 \nabla_{\mathbf{c}_1} f_1 d\mathbf{c}_1 = -\nabla \cdot \Psi \quad (31)$$

where  $I = -\nabla \cdot \Psi$ .

In the case when  $\phi_1$  is not associated to a microscopic conservation law the results are

$$I_0^{SET} = I_0^{RET} = \frac{4\chi_c}{\pi \mathbf{Kn}} \int (\phi'_1 - \phi_1) f_1 f_2 (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}_2, \quad (32)$$

$$I_1^{SET} = I_1^{RET} = \frac{4}{\pi \mathbf{Kn}} \int (\phi_1 - \phi'_1) \left[ \chi_c \mathbf{k} \cdot f_1 \nabla f_2 + \frac{1}{2} (\mathbf{k} \cdot \nabla_{\mathbf{r}} \chi_c) f_1 f_2 \right] \times (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}_2, \quad (33)$$

$$I_2^{SET} = \frac{2}{\pi \mathbf{Kn}} \int (\phi'_1 - \phi_1) \left[ (\mathbf{k} \cdot \nabla \chi_c) \mathbf{k} (f_1 \nabla f_2) + \chi_c \mathbf{k} \mathbf{k} : f_1 \nabla \nabla f_2 + \frac{1}{4} (\mathbf{k} \mathbf{k} : \nabla \nabla \chi_c) f_1 f_2 \right] (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}_2, \quad (34)$$

$$I_2^{RET} = \frac{2}{\pi \mathbf{Kn}} \int (\phi'_1 - \phi_1) \left[ (\mathbf{k} \cdot \nabla \chi_c) \mathbf{k} (f_1 \nabla f_2) + \chi_c \mathbf{k} \mathbf{k} : f_1 \nabla \nabla f_2 + \frac{1}{4} \frac{\partial^2 \chi_c}{\partial n^2} (\mathbf{k} \nabla n(\mathbf{r}, t)) (\mathbf{k} \nabla n(\mathbf{r}, t)) f_1 f_2 \right] (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}_2. \quad (35)$$

## 5. SIMPLIFIED COLLISION CONTRIBUTION

The collisional term in both Boltzmann's and Enskog's equations is quadratic in the distribution function, hence there are integrals which involve products  $H^\ell H^m$  which have to be projected to different  $H^n$ s. Even in the case of the 13 moments method for Boltzmann's equation with the contraction as in Eq. (11), Grad proposed to consider only those contributions such that  $\ell + m = n$  even though there is no clear mathematical criterion to make such simplification except for Maxwellian molecules because in this case the collision term is integrable and such quadratic terms do not appear.

Nevertheless, Grad was able to prove that very little error is made even in the case of non-Maxwellian molecules by restricting the contributions to  $\ell + m = n$ .<sup>(11,12)</sup> Although we do not have such a prove in the case of Ebskog's equation we follow the same prescription.

In our present case, to be able to obtain general hydrodynamic equations—following the present procedure—which are not overwhelmingly complex it is necessary to make some further simplifications.

Since  $\Phi_G$  is a correcting factor we may consider that its contribution can be regarded as 1 plus something small, schematically  $\Phi_G = 1 + \varepsilon$ . Since the collision integral is quadratic in the distribution function, the integrand has a product  $\Phi_{G1} \Phi_{G2} = 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1 \varepsilon_2$ . Our simplificatory strategy will be first

that: to evaluate  $J_0$  we keep all the contributions; to evaluate  $J_1$  we keep the "order  $\varepsilon$ " terms, namely,  $\Phi_{G1}\Phi_{G2} \approx \Phi_{G1} + \Phi_{G2} - 1$ ; to evaluate  $J_2$  we take  $\Phi_{G1}\Phi_{G2} \approx 1$ .

Therefore, in Eq. (33), the following relations are required,

$$f_1 f_2 = f_{M1} f_{M2} (\Phi_{G1} + \Phi_{G2} - 1),$$

$$f_1 \nabla f_2 = f_{M1} (\nabla f_{M2}) (\Phi_{G1} + \Phi_{G2} - 1) + f_{M1} f_{M2} \nabla \Phi_{G2}$$

and in Eqs. (34) and (35) we use

$$f_1 f_2 = f_{M1} f_{M2}. \tag{36}$$

For hard disks,  $f_{M1} f_{M2} = f'_{M1} f'_{M2}$ . Hence,  $J_{24}$  in Eq. (25) vanishes, and from Eq. (24), the SET's collision term coincides with the RET's. Additionally, also  $J_{23}$  in Eq. (25) vanishes for the same reason. Consequently, Eq. (35) is reduced (34) and

$$I_2^{SET} = I_2^{RET} = \frac{2}{\pi \text{Kn}} \int (\phi'_1 - \phi_1) [(\mathbf{k} \cdot \nabla \chi_c) \mathbf{k} (f_{M1} \nabla f_{M2}) + \chi_c \mathbf{k} \mathbf{k} : f_{M1} \nabla \nabla f_{M2}] (\mathbf{g} \cdot \mathbf{k}) \theta_{\mathbf{k}} d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}_2. \tag{37}$$

After introducing the approximations mentioned above, we directly calculate, using MAPLE routines, the collisional term in Enskog's equation for all contributions originated in the products  $H^l H^m$ .

## 6. THE HYDRODYNAMIC EQUATIONS

The central result of the present paper are the hydrodynamic equations that we have derived from Enskog's equation according to what has been explained in the previous sections.

To the best of our knowledge, these hydrodynamic equations are published for the first time. For the two dimensional case we have not encountered any paper at all while for three dimensional case the published literature refers to linearized versions. For the sake of clarity the balance equations for conserved quantities (mass, momentum and energy) are written in terms of the hydrodynamic quantities: density, hydrodynamic velocity, temperature, pressure tensor and heat flux; whereas the balance equations associated to the pressure tensor and the heat flux are written in terms of the symmetric pressure tensor and heat flux vector. This is what is also done in almost all well-known articles for the extremely dilute system.

We use the following notation as Euler's (material) derivative:  $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}}$ . Summation is implied with repeated tensorial indices.

The final extended hydrodynamic equations are the following:

$$\frac{Dn}{Dt} = -n\nabla \cdot \mathbf{v} \quad (38)$$

$$n \frac{D\mathbf{v}}{Dt} = n\mathbf{F} - \nabla \cdot \mathbf{P} \quad (39)$$

$$n \frac{DT}{Dt} = -(\nabla \cdot \mathbf{Q} + \mathbf{P} : \nabla \mathbf{v}) \quad (40)$$

$$\begin{aligned} \frac{D}{Dt} p_{ij} + \frac{\partial v_l}{\partial x_l} p_{ij} + p_{il} \frac{\partial v_l}{\partial x_i} + p_{jl} \frac{\partial v_l}{\partial x_j} - p_{lm} \frac{\partial v_m}{\partial x_l} \delta_{ij} \\ + \frac{1}{2} \left[ \frac{\partial}{\partial x_i} q_j^k + \frac{\partial}{\partial x_j} q_i^k - \frac{\partial}{\partial x_l} q_l^k \delta_{ij} \right] + nT S_{ij} = \mathbf{I}_p \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{D}{Dt} q_i^k + \frac{3}{2} \left[ \frac{\partial v_l}{\partial x_l} q_i^k + \frac{\partial v_l}{\partial x_i} q_l^k \right] + \frac{1}{2} \frac{\partial v_l}{\partial x_i} q_l^k \\ + T \frac{\partial}{\partial x_l} p_{li} + 2 \frac{\partial T}{\partial x_l} p_{li} - \frac{1}{n} \frac{\partial}{\partial x_m} P_{ml}^k p_{li} \\ + 2nT \frac{\partial T}{\partial x_i} = \mathbf{I}_q \end{aligned} \quad (42)$$

where the pressure tensor  $\mathbf{P}$  is given as

$$\mathbf{P} = \beta_p \mathbf{p} + \mathbf{P}_{\text{col}}, \quad (43)$$

with

$$\beta_p = 1 + \frac{\delta}{\text{Kn}} \chi_c n, \quad (44)$$

$$\mathbf{P}_{\text{col}} = \left[ 1 + \delta \frac{2}{\text{Kn}} \chi_c n \right] nT \mathbf{1} - \frac{\delta^2}{\sqrt{\pi} \text{Kn}} \chi_c n^2 \sqrt{T} [\mathbf{S} + 2(\nabla \cdot \mathbf{v}) \mathbf{1}] \quad (45)$$

and the shear tensor  $\mathbf{S}$  which is defined by

$$S_{ij} = \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{\partial v_l}{\partial x_l} \delta_{ij}, \quad (46)$$

where  $\mathbf{Q} = \beta_q \mathbf{q}^k + \mathbf{q}_{\text{col}}$ ,

$$\beta_q = 1 + \delta \frac{3}{2\text{Kn}} \chi_c n, \quad \mathbf{q}_{\text{col}} = -\delta^2 \frac{2}{\sqrt{\pi} \text{Kn}} \chi_c n^2 \sqrt{T} \nabla T \quad (47)$$

and where the kinetic contribution to the heat flux vector  $\mathbf{Q}$  is given by  $\mathbf{q}^k$  following the definition Eq. (12). Additionally, the remaining terms, which give the collisional contribution to the extended hydrodynamic equations, are given as below.

$\mathbf{I}_p$  is split in the form  $\mathbf{I}_p = \mathbf{I}_p^0 + \mathbf{I}_p^1 + \mathbf{I}_p^2$  where

$$\mathbf{I}_p^0 = -\frac{8}{\sqrt{\pi}\text{Kn}}\chi_c\sqrt{T}\left[ np_{ij} + \frac{1}{128T^2}\left[ 2q_i^k q_j^k - q_l^k q_l^k \delta_{ij} \right] \right], \quad (48)$$

$$\begin{aligned} \mathbf{I}_p^1 = & -\frac{\delta}{\text{Kn}}\left[ \frac{5}{4}\chi_c\left[ q_i^k \frac{\partial n}{\partial x_j} + q_j^k \frac{\partial n}{\partial x_i} - q_l^k \frac{\partial n}{\partial x_l} \delta_{ij} \right] \right. \\ & + \frac{3}{4}\chi_c\left[ \frac{\partial}{\partial x_i} q_j^k + \frac{\partial}{\partial x_j} q_i^k - \frac{\partial}{\partial x_l} q_l^k \delta_{ij} \right] \\ & \left. + n\left[ q_i^k \frac{\partial \chi_c}{\partial x_j} + q_j^k \frac{\partial \chi_c}{\partial x_i} - q_l^k \frac{\partial \chi_c}{\partial x_l} \delta_{ij} \right] - \chi_c \frac{\partial v_l}{\partial x_l} p_{ij} + \frac{5}{4}\chi_c n^2 T S_{ij} \right] \quad (49) \end{aligned}$$

and the  $\delta^2$  contribution is

$$\begin{aligned} \mathbf{I}_p^2 = & \frac{\delta^2}{\text{Kn}}\frac{1}{\sqrt{\pi}}\left[ 2\chi_c n \sqrt{T}\left[ \frac{\partial n}{\partial x_i} \frac{\partial T}{\partial x_j} + \frac{\partial n}{\partial x_j} \frac{\partial T}{\partial x_i} - \frac{\partial n}{\partial x_l} \frac{\partial T}{\partial x_l} \delta_{ij} \right] \right. \\ & + \frac{1}{2}\chi_c \frac{n^2}{\sqrt{T}}\left[ 2\frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_j} - \frac{\partial n}{\partial x_l} \frac{\partial T}{\partial x_l} \delta_{ij} \right] \\ & + \chi_c n^2 \sqrt{T}\left[ 2\frac{\partial^2 T}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_l \partial x_l} T \right] \\ & + n^2 \sqrt{T}\left[ \frac{\partial T}{\partial x_i} \frac{\partial \chi_c}{\partial x_j} + \frac{\partial T}{\partial x_j} \frac{\partial \chi_c}{\partial x_i} - \frac{\partial T}{\partial x_l} \frac{\partial \chi_c}{\partial x_l} \delta_{ij} \right] \\ & \left. + 2\chi_c n^2 \sqrt{T} \frac{\partial v_l}{\partial x_l} S_{ij} \right], \quad (50) \end{aligned}$$

$\mathbf{I}_q$  is split  $\mathbf{I}_q = \mathbf{I}_q^0 + \mathbf{I}_q^1 + \mathbf{I}_q^2$  and

$$\mathbf{I}_q^0 = -\frac{\chi_c}{\sqrt{\pi T}\text{Kn}}\left[ 4nTq_i^k + p_{il}q_l^k \right], \quad (51)$$

$$\mathbf{I}_q^1 = \frac{\delta}{\text{Kn}}\left[ (1)_\delta + (2)_\delta + (3)_\delta \right]$$

$$\begin{aligned} (1)_\delta = & -\frac{1}{2}\left[ 7\chi_c n T \frac{\partial}{\partial x_l} p_{li} + 7T\chi_c \frac{\partial n}{\partial x_l} p_{li} \right. \\ & \left. + 7nT \frac{\partial \chi_c}{\partial x_l} p_{li} + 4\chi_c \frac{\partial T}{\partial x_l} p_{li} \right], \end{aligned}$$

$$(2)_\delta = - \left[ 8\chi_c n T^2 \frac{\partial n}{\partial x_i} + 7\chi_c n^2 T \frac{\partial T}{\partial x_i} + 4n^2 T^2 \frac{\partial \chi_c}{\partial x_i} \right],$$

$$(3)_\delta = -\frac{1}{2}\chi_c n \left[ 2S_{il} - \frac{\partial v_l}{\partial x_l} \delta_{ij} \right] q_i^k \quad (52)$$

and where

$$\mathbf{I}_q^2 = \frac{\delta^2}{8\text{Kn}\sqrt{\pi}} n \sqrt{T} \left[ 35\chi_c n^2 \frac{\partial T}{\partial x_l} S_{li} + 18nT \frac{\partial \chi_c}{\partial x_l} S_{li} \right. \\ \left. + 36\chi_c T \frac{\partial n}{\partial x_l} S_{li} + 18\chi_c n T \frac{\partial^2}{\partial x_l \partial x_l} v_i + 36\chi_c n T \frac{\partial}{\partial x_l} \frac{\partial v_l}{\partial x_l} \right]. \quad (53)$$

The above equations have the well known form which comes from very general arguments from continuous fluid matter and where established in the XIX century. The detail structure of some of the terms comes from kinetic theory.

Of course it is interesting to compare our results to the ones obtained by means of the Chapman–Enskog method. For example, Kim and Hayakawa<sup>(26)</sup> have derived the explicit velocity distribution function of Boltzmann’s equation using Chapman–Enskog expansions to Burnett’s level, although restricted to steady states. Also the linearized Burnett equations in the SET (in the first Enskog approximation) have been used by Alves and Kremer<sup>(32)</sup> to study light scattering from density fluctuations. However it would be difficult to compare our approximations with those from Chapman–Enskog’s method. They are quite different methods indeed.

Furthermore, in order to calculate any physical quantity, we first determine  $\chi_c$ . The static pressure  $p$  is given as  $p = \frac{1}{2} \text{Tr} \mathbf{P}$ .

The static pressure  $p$  is given as  $p = \frac{1}{2} \text{Tr} \mathbf{P}$ . This automatically yields that the equation of state is given by Eqs. (43) and (44). Then, the relation between  $\chi_c$  and the static pressure  $p$  is given by

$$\chi_c = \frac{p/nT - 1}{2\rho_0} \quad (54)$$

where the relation:  $\delta = \rho_0 \text{Kn}$  is used. In general,  $\chi_c$  is related to the pair distribution function.<sup>(4)</sup> In the case without external force, the equation of state at equilibrium is very well approximated using Henderson’s expression<sup>(33)</sup>

$$\frac{p}{nT} = \frac{1 + \frac{\rho_0^2}{8}}{(1 - \rho_0)^2}, \quad (55)$$

which is what we use.

In the case with  $F \neq 0$ , SET does not yield the correct single-particle equilibrium distribution function, whereas RET do.<sup>(34)</sup>

## 7. HEAT CONDUCTION BETWEEN PARALLEL PLATES UNDER A STEADY STATE CONDITION

### 7.1. Reduced Hydrodynamics

The above hydrodynamics should in principle allow us to study a wide variety of dynamic problems involving dense gases. In this section we look at a quite simple situation in this wide context. It is the case of a one dimensional heat conduction case between two parallel plates.

The system consists of gas between two infinite parallel plates 1 and 2 separated by a distance  $L$ . The plates have fixed temperatures  $T_1$  and  $T_2$  respectively. A schematic representation in Fig. 1 shows the  $Y$  axis defined perpendicular to the plates while the  $X$  axis is placed on plate 1.

In general, under circumstances with a gradient of temperature perpendicular to the walls, there is a difference between the temperature of the gas by the plate and the temperature of the plate itself. This is a well-known effect called thermal slip. For simplicity's sake we neglect such difference.

For this simple system the extended hydrodynamic equations drastically simplify. The basic concrete equations solved here are the following:

$$P_{yy}(y) \equiv P_{yy} = \text{const.}, \quad Q_y(y) \equiv Q_y = \text{const.},$$

$$P_{xy}(y) = p_{xy}(y) = 0,$$

$$P_{yy} = - \left[ 1 + \frac{\delta}{\text{Kn}} \chi_c n(y) \right] p_{xx}(y) + \left[ 1 + 2 \frac{\delta}{\text{Kn}} \chi_c n(y) \right] n(y) T(y), \quad (56)$$

$$Q_y = \left[ 1 + \frac{3}{2} \chi_c n(y) \frac{\delta}{\text{Kn}} \right] q_y^k(y) - \delta^2 \frac{2}{\sqrt{\pi} \text{Kn}} \chi_c n(y)^2 \sqrt{T(y)} \frac{dT(y)}{dy}, \quad (57)$$

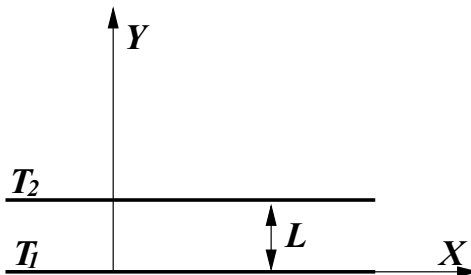


Fig. 1. Configuration of the simple heat conduction system as described in the text.

$$\begin{aligned}
-\frac{1}{2} \frac{dq_y^k(y)}{dy} = & -\frac{8}{\sqrt{\pi} \text{Kn}} \chi_c \sqrt{T(y)} \left[ n(y) p_{xx}(y) - \frac{q_y^k(y)^2}{128 T(y)^2} \right] \\
& + \frac{\delta}{4 \text{Kn}} \chi_c \left[ 5q_y^k(y) \frac{dn(y)}{dy} + 3n(y) \frac{dq_y^k(y)}{dy} \right] \\
& - \frac{\delta^2}{\sqrt{\pi} \text{Kn}} \chi_c n(y) \sqrt{T(y)} \left[ 2 \frac{dn(y)}{dy} \frac{dT(y)}{dy} + \frac{1}{2} \frac{n(y)}{T(y)} \left( \frac{dn(y)}{dy} \right)^2 \right. \\
& \left. + n(y) \frac{d^2 T(y)}{dy^2} \right], \tag{58}
\end{aligned}$$

$$\begin{aligned}
& -T(y) \frac{dp_{xx}(y)}{dy} - 2 \frac{dp_{xx}(y)}{dy} \frac{dT(y)}{dy} + 2n(y)T(y) \frac{dp_{xx}(y)}{dY} \\
& + \frac{T(y)}{n(y)} p_{xx}(y) \frac{dT(y)}{dy} - \frac{p_{xx}(y)}{n(y)} \frac{dp_{xx}(y)}{dy} = -\frac{4}{\sqrt{\pi} T(y) \text{Kn}} \\
& \times [n(y)T(y) - p_{xx}(y)] q_y^k(y) + \frac{\delta}{\text{Kn}} \chi_c \\
& \times \left[ \frac{7}{2} n(y)T(y) \frac{dp_{xx}(y)}{dy} + \frac{7}{2} T(y) p_{xx}(y) \frac{dn(y)}{dY} + 2p_{xx}(Y) \frac{dT(y)}{dy} \right. \\
& \left. - 8n(y)T(y)^2 \frac{dn(y)}{dy} - 7n(Y)^2 T(y) \frac{dT(y)}{dy} \right]. \tag{59}
\end{aligned}$$

## 7.2. Functional Expression of the Pressure

All the results we describe in what follows were obtained using perturbation methods choosing  $T_2 > T_1$  and using  $\varepsilon = (T_2 - T_1)/T_1$  as perturbation parameter. We solve the system of equations up to  $\varepsilon^6$  and try to express the nonequilibrium pressure  $P_{yy}$  in terms of the density, temperature and the heat flux  $Q_y$ . Note that here  $P_{yy}$  is constant. We manage to do this up to third order in  $\varepsilon$ , which corresponds to situations not very far from equilibrium. The functional expression for the nonequilibrium pressure  $P_{yy}$  turns out to be

$$\begin{aligned}
P_{yy}[n(y), T(y), Q_y] = & n(y)T(y) \left[ 1 + 2\chi_c \frac{\delta}{\text{Kn}} n(y) \right] \\
& \times \left[ 1 + \lambda_p \frac{Q_y^2}{n(y)^2 T(y)^3} \Lambda[n(y), T(y)] \right] \tag{60}
\end{aligned}$$



where  $\lambda_p$  is the following constant

$$\begin{aligned} \lambda_p = & -\frac{1}{128} + \frac{7}{16}\chi_c \frac{\delta}{\text{Kn}} - \frac{1}{128} \left[ \frac{9}{\pi} + 2253 \right] \chi_c^2 \left[ \frac{\delta}{\text{Kn}} \right]^2 \\ & + \left[ \frac{7}{32}\chi_c \frac{\delta}{\text{Kn}} - \frac{1}{128} \left[ \frac{9}{\pi} + 2309 \right] \chi_c^2 \left[ \frac{\delta}{\text{Kn}} \right]^2 \right] \varepsilon \\ & + \left[ \left[ -\frac{7}{384} + \frac{7\pi \text{Kn}^2}{16384\chi_c^2} \right] \chi_c \frac{\delta}{\text{Kn}} - \frac{1}{128} \left[ \frac{2356}{6} + \frac{3}{2\pi} - \frac{3 + 751\pi \text{Kn}^2}{256\chi_c^2} \right] \right. \\ & \times \chi_c^2 \left[ \frac{\delta}{\text{Kn}} \right]^2 \left. \right] \varepsilon^2 + \left[ \left[ \frac{7}{768} + \frac{7\pi \text{Kn}^2}{16384\chi_c^2} \right] \chi_c \frac{\delta}{\text{Kn}} - \frac{1}{128} \left[ \frac{3 + 359\pi}{256} \right] \delta^2 \right] \varepsilon^3 \end{aligned} \tag{61}$$

and where

$$\begin{aligned} \Lambda[n(y), T(y)] = & 1 + n(y)(1 + 56T(y))\chi_c \frac{\delta}{\text{Kn}} \\ & + \left[ -\frac{32 + 305\pi}{4\pi} + 784T(y) + \frac{3(257\pi - 3)}{\pi} T(y)^2 \right]. \end{aligned} \tag{62}$$

Substituting  $\delta = 0$  in Eq. (56) yields the functional expression for the pressure  $P_{yy}$ ,

$$P_{yy}(n, T, Q_y) = n(y)T(y) \left[ 1 - \frac{1}{128} \frac{Q_y^2}{n(y)^2 T(y)^3} \right]. \tag{63}$$

Therefore Eq. (60) is a natural extension of Eq. (63). For hard spheres the expression which correspond to Eq. (63) was already given in Ref. 11. For hard disks Eq. (63) is given for the first time but it can be obtained from the appendix in Ref. 35,<sup>2</sup> where a gas of *inelastic* disks is studied.

### 7.3. Comment

The steady state thermodynamics (SST) proposed by Sasa and Tasaki<sup>(29)</sup> assumes that all nonequilibrium variables and thermodynamic functions can be expressed in terms of the density, temperature and the steady state fluxes (heat flux in the case of our example). We found that using Grad's moment method,<sup>(11,12)</sup> this

<sup>2</sup>These hydrodynamic equations given in this paper are for granular gases. But substituting  $q = 0$  to these equations, the hydrodynamic equations corresponding to Boltzmann's gas is given. Actually, for hard disks it is the first time that the hydrodynamic equations with the method of Grad starting from Boltzmann equation appear. And from it, in steady state case which is investigated in present paper, Eq. (63) is given automatically.

condition is satisfied in the simple heat conductive case for the dilute and dense gases which obey Boltzmann's or Enskog's kinetic equations in the range of the approximation introduced in Sec. 5 if the system is not too far from equilibrium. Unfortunately, because kinetic theory defines only the internal energy density,  $u(y) \propto T(y)$ , and this quantity is interpreted as the temperature, it is impossible, within the framework of kinetic theory alone, to construct nonequilibrium thermodynamic functions like the Helmholtz energy. Using Boltzmann's entropy:  $s = -f \ln f$  ( $s$  is the entropy density), one can define the entropy and a temperature through  $(\partial s / \partial u) = 1/\theta$  where  $\theta$  is the nonequilibrium temperature<sup>(36)</sup> in what is called *extended irreversible thermodynamics* (EIT).<sup>(36)</sup> Furthermore one could define the nonequilibrium Helmholtz free energy and chemical potential, although we have not seen anyone developing such a formalism. However within kinetic theory it is not clear which is the difference between the temperature and the internal energy density. Even if the entropy of the system were given it is not possible to determine Helmholtz free energy and therefore it is impossible to determine the chemical potential. Using an assumption of SST<sup>(29)</sup> that the Maxwell relation is satisfied by the nonequilibrium thermodynamic variables, then from Eq. (60) a chemical potential follows without having to determine Helmholtz's free energy, although this chemical potential still has an undetermined part which depends on the heat flux and temperature. In a following article<sup>(37)</sup> we propose a method to get the chemical potential partially using the framework of SST.<sup>(29)</sup> The undetermined part is measured experimentally. By this method, one can determine the value of the chemical potential at the interface between equilibrium and nonequilibrium sides. In other words, we will propose an experimental method for determining the values of the nonequilibrium variables at the equilibrium-nonequilibrium interface following the Sasa and Tasaki scheme. Once a chemical potential is given it is possible to completely express the condition that the perfect  $\mu$  wall must have<sup>(29)</sup> and to discuss about the prediction of<sup>(29)</sup> beyond any doubt.

## 8. SUMMARY AND FINAL REMARKS

In this article we have given in detail the hydrodynamic equations for a bi-dimensional dense gas of hard disks that follow from Enskog's equation using Grad's moment expansion method. Bearing in mind that without a strategy to simplify the calculations no manageable hydrodynamics can be obtained we have introduced in Sec. 5 a simplifying scheme that leads to a hydrodynamics which is more complete than the one that would follow from a linear approximation scheme. As far as we know this is the first time that Grad's method used to obtain extended hydrodynamic equations from Enskog's equation has been published beyond the a linear approximation in two or three dimensions.

We have applied this hydrodynamic equations to discuss a simple but subtle 1D case of heat conduction. The example has the virtue of giving the opportunity

to discuss the adequacy of the starting point of SST.<sup>(29)</sup> We have shown that this simple non-equilibrium steady state system, not too far from equilibrium, behaves as if nonequilibrium thermodynamic variables could be expressed in a functional form. Hence, it may be possible to construct a nonequilibrium thermodynamics without any information of the microscopic dynamics of the system. First, we have shown this for the dense hard disk system. It is well known that for the hydrodynamics obtained from Boltzmann's equation using Grad's method this is correct.<sup>(11,12)</sup>

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